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A SIMPLICIAL FORMULA AND BOUND FOR THE EULER CLASS

BY

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ABSTRACT

Let G be the discrete group of orientation preserving diffeomorphisms of the circle. An explicit simplicial formula on the level of the bar construction is given for the Euler Class of a circle bundle with structure group G. An upper bound for the Euler Class is obtained which, when the base space of the bundle is a closed orientable surface, reduces to that of J. Wood. An invariant of circle bundles, complexity, is defined which "detects" the upper bound.

Let G^r be the group of orientation preserving C^r diffeomorphisms of the circle S^1 and let H^r be the subgroup of those which are the identity in a neighborhood of a given point of S^1. For any discrete group G let $H_*(G)$ denote the integral homology of BG, that is $H_*(G) = H_*(K(G, 1), Z)$. In [3] I proved that there is a short exact sequence for $0 \leqq r \leqq \infty$, $r \neq 2$,

$$0 \to H_2(H^r) \to H_2(G^r) \xrightarrow{E} Z \to 0.$$

In this paper I will

(1) give an explicit formula for E on the chain level of the bar construction on G^r,
(2) identify E with the classical Euler Class,
(3) use the formula in (1) to obtain an upper bound for the Euler Class of a circle bundle with structure group G on any given CW complex, and
(4) define an invariant of a circle bundle, the complexity, which detects the upper bound.

My intention in this paper is to present a completely simplicial treatment of the Euler Class. All the analysis is very elementary and carried out on the level

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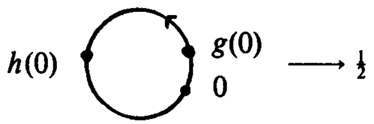
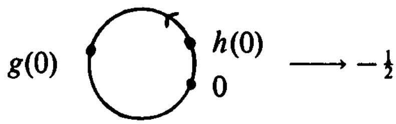
of the simplicial nerve of G. The formula given for E in §1 assigns $0, \frac{1}{2}$ or $-\frac{1}{2}$ to every 2-simplex (g, h) of the nerve of G according to an "ordering" of $(g(0), h(0))$:





with all other configurations of $(g(0), h(0))$ given the value 0 (formula $*$). Any cycle turns out to be integer valued (Proposition 1) and the above assignment is well defined on homology (Proposition 2).

Recall the theorem of J. Wood, [7]. If X is a closed orientable 2-manifold, $[X]$ its fundamental class and $h : X \to BG$ any map, then $-|Eh_*[X]|$ is greater than the Euler characteristic of X, $\varepsilon(X)$. The bound given for E in Theorems 4 and 5 of this paper is a simplicial version of Wood's theorem.

For any space X and for any $\alpha \in H_2(X)$ I define an integer $\kappa(\alpha)$ called the complexity of α which is roughly the number of 2-cells needed to construct α. Then for any map $h : X \to BG$ the following inequality must hold (Theorem 4):

$$|Eh_*\alpha| \leqq \kappa(\alpha)/2. \tag{1}$$

The right-hand side is independent of h so gives an obstruction to the existence of h. That such a bound exists will be seen to be a very simple consequence of the fact that E can be defined simplicially on the nerve of G.

The number $\kappa(\alpha)$ is the integer analogue of Gromov's norm $\| \ \|$ on $H_*(X, \mathbf{R})$ [2]. Over the reals the bound becomes

$$|Eh_*\alpha| \leqq \|\alpha\|/2 \tag{2}$$

for all $\alpha \in H_2(X, \mathbf{R})$.

If X is a closed surface this reduces exactly to Wood's inequality. For take $\alpha = [X]$. Then $\|X\| = -2\varepsilon(X)$ (see [2]), and (2) becomes $-|Eh_*[X]| \geqq \varepsilon(X)$. So this is the best available bound which would work for spaces.

We finally reformulate our theorem in a way that makes more sense over general spaces, for it does not depend on a choice of α. Consider the affine subspace $Eh_*^{-1}(1) \subset H_2(X, \mathbf{R})$. Define the *complexity of a bundle γ over X, $\kappa(\gamma)$*, to be the inf of the norms of all the elements in this space. I will show (and it is an easy consequence of (2)) that if γ reduces to a bundle with discrete structure group then $\kappa(\gamma) \geqq 2$.

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1. An invariant $e : H_2(G) \to Z$

Let G stand for any of the groups G^r. Recall the construction of the simplicial nerve of G. $NG : k \to N_kG$ is a simplicial set with k cells $(g_1, \ldots, g_k)$, $g_i \in G$, $k \geqq 1$ and one 0-cell. The faces are given by

$$\partial_0(g_1, \ldots, g_k) = (g_1^{-1}g_2, \ldots, g_1^{-1}g_k);$$
$$\partial_i(g_1, \ldots, g_k) = (g_1, \ldots, \hat{g}_i, \ldots, g_k), \quad 1 \leqq i \leqq k.$$

The degeneracies $\sigma_i : N_kG \to N_{k+1}G$ are given by

$$\sigma_i(g_1, \ldots, g_k) = (g_1, \ldots, g_i, g_i, \ldots, g_k).$$

$|NG| = BG = K(G, 1)$ so the homology $H_*(G)$ is given by the homology of the simplicial set NG. This is the classical bar construction.

Now consider a pair $(g, h) \in G \times G$. Lift g and h to orientation preserving periodic diffeomorphisms of $\mathbf{R}$, $\tilde{g}$, $\tilde{h}$ so that $\tilde{g}(0) \in [0, 1)$ and $\tilde{h}(0) \in [0, 1)$.

FORMULA $*$. Set

$$\begin{cases} e(g, h) = \frac{1}{2} & \text{if } 0 < \tilde{g}(0) < \tilde{h}(0), \\ e(g, h) = -\frac{1}{2} & \text{if } 0 < \tilde{h}(0) < \tilde{g}(0), \\ e(g, h) = 0 & \text{in all other cases.} \end{cases}$$

Extend e to a homomorphism from $C_2(G)$ = free abelian group on $G \times G$ to $\mathbf{Q}$. A cycle $z \in Z_2(G) \subset C_2(G)$ is given by $z = \Sigma_{i=1}^n (g_i, h_i)$ satisfying

$$\sum_{i=1}^{n} (g_1^{-1}h_i - h_i + g_i) = 0.$$

A boundary $w \in B_2(G)$ is a sum of cycles of the form

$$(g^{-1}h, g^{-1}k) - (h, k) + (g, k) - (g, h)$$

and

$$H_2(G) = Z_2(G)/B_2(G).$$

THEOREM 1. *There is a well defined homomorphism* $e : H_2(BG) \to Z$ *defined by* $*$.

This will follow from

PROPOSITION 2. *For* $z \in Z_2(G)$, $e(z)$ *is an integer*.

PROPOSITION 3. *For* $z \in B_2(G)$, $e(z) = 0$.

To verify Proposition 1 first note that if $z = \Sigma_{i=1}^{n} (g_i, h_i)$ is a cycle, then n is even. If $e(g_i, h_i) = \pm \frac{1}{2}$ for a given i then *none* of the faces $g_i^{-1}h_i$, g_i or h_i has a fixed point at $[0] \in \mathbf{R}/Z = S^1$. On the other hand, if $e(g_i, h_i) = 0$ either one or all three (in any case an odd number) of faces has a fixed point at $[0] \in S^1$. Since the faces with fixed points must cancel among themselves, the number of (g_i, h_i) with $e(g_i, h_i) = 0$ must be even. So the number of (g_i, h_i) with $e(g_i, h_i) = \pm \frac{1}{2}$ must also be even, proving the first proposition.

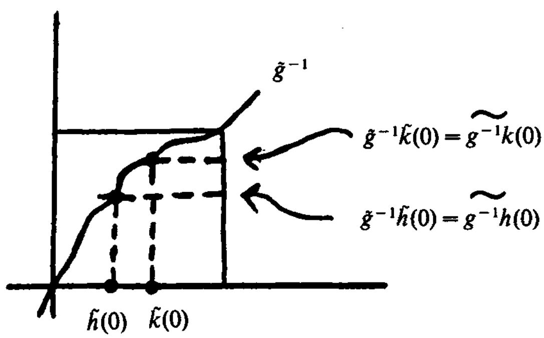
The fact that the invariant e does not bound, i.e. that Proposition 2 is true, can be deduced from results in [3]. But to make the construction of e self-contained I will prove Proposition 2 directly.

To prove Proposition 2 it must be shown that for all g, h, k

$$(e[(g^{-1}h, g^{-1}k) - (h, k) + (g, k) - (g, h)]) = 0.$$

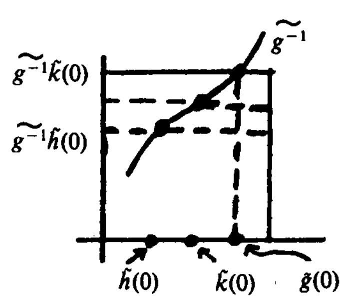
There are several cases to verify. I will check the typical ones and leave the rest to the reader. By then the essential points will be evident.

Suppose that $0 < \tilde{h}(0) < \tilde{k}(0)$. If $g^{-1}(0) = 0$ then $e(g^{-1}h, g^{-1}k) = \frac{1}{2}$, $e(h, k) = \frac{1}{2}$, $e(g, k) = 0$ and $e(g, h) = 0$ so that $e = 0$.

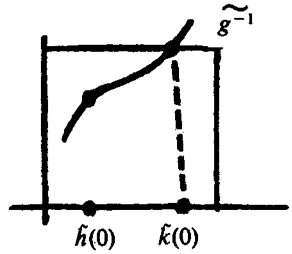


So assume $\widetilde{g^{-1}}(0) \neq 0$. Then $\widetilde{g^{-1}}\tilde{h}(0)$ and $\widetilde{g^{-1}}\tilde{k}(0)$ are related in one of the following ways:

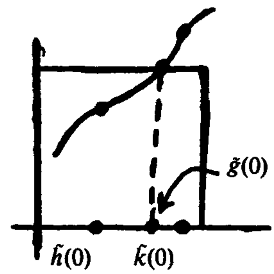
(a) $0 < \widetilde{g^{-1}}\tilde{h}(0) < \widetilde{g^{-1}}\tilde{k}(0) < 1$



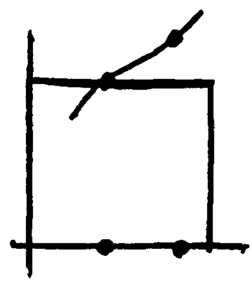
(b) $0<\widetilde{g^{-1}}\tilde{h}(0)<\widetilde{g^{-1}}\tilde{k}(0)=1$



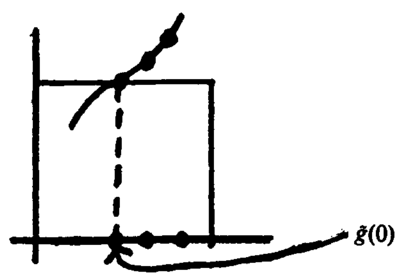
(c) $0<\widetilde{g^{-1}}\tilde{h}(0)<\widetilde{g^{-1}}\tilde{k}(0)<2$



(d) $1=\widetilde{g^{-1}}\tilde{h}(0)<\widetilde{g^{-1}}\tilde{k}(0)<2$



(e) $1<\widetilde{g^{-1}}\tilde{h}(0)<\widetilde{g^{-1}}\tilde{k}(0)<2$



The following equations and inequalities and corresponding values of e are evident from the above graphs.

Case (a): $0<\widetilde{g^{-1}h}(0)<\widetilde{g^{-1}k}(0)\rightarrow e=\frac{1}{2}$

$0<\tilde{h}(0)<\tilde{k}(0) \rightarrow e=\frac{1}{2}$

$0<\tilde{k}(0)<\tilde{g}(0) \rightarrow e=-\frac{1}{2}$

$0<\tilde{h}(0)<\tilde{g}(0) \rightarrow e=-\frac{1}{2}.$

So $e = 0$ in this case.

Case (b): $\widetilde{g^{-1}k}(0) = 0 \quad \to e = 0$
$0 < \tilde{h}(0) = \tilde{k}(0) \to e = \frac{1}{2}$
$\tilde{g}(0) = \tilde{k}(0) \quad \to e = 0$
$\tilde{h}(0) < \tilde{g}(0) \quad \to e = -\frac{1}{2}$.

So $e = 0$ in this case.

Case (c): $0 < \widetilde{g^{-1}k}(0) < \widetilde{g^{-1}h}(0) \to e = -\frac{1}{2}$
$0 < \tilde{h}(0) < \tilde{k}(0) \quad \to e = \frac{1}{2}$
$0 < \tilde{g}(0) < \tilde{k}(0) \quad \to e = \frac{1}{2}$
$0 < \tilde{h}(0) < \tilde{g}(0) \quad \to e = -\frac{1}{2}$.

So $e = 0$ in this case.

Case (d): $\widetilde{g^{-1}h}(0) = 0 \quad \to e = 0$
$0 < \tilde{h}(0) < \tilde{k}(0) \to e = \frac{1}{2}$
$0 < \tilde{g}(0) < \tilde{k}(0) \to e = \frac{1}{2}$
$\tilde{g}(0) = \tilde{h}(0) \quad \to e = 0$.

So $e = 0$ in this case.

Case (e): $0 < \widetilde{g^{-1}h}(0) < \widetilde{g^{-1}k}(0) \to e = \frac{1}{2}$
$0 < \tilde{h}(0) < \tilde{k}(0) \quad \to e = \frac{1}{2}$
$0 < \tilde{g}(0) < \tilde{k}(0) \quad \to e = \frac{1}{2}$
$0 < \tilde{g}(0) < \tilde{h}(0) \quad \to e = \frac{1}{2}$.

Now if instead $0 < \tilde{k}(0) < \tilde{h}(0)$, an exact case by case analysis again shows $e = 0$.

Finally, the cases $\tilde{h}(0) = \tilde{k}(0)$, and $\tilde{h}(0) = 0$ or $\tilde{k}(0) = 0$ can all be verified by the same procedure as above.

2. Identification of the invariant e with the Euler Class

In this section I will explicitly construct a cycle $z \in Z_2(BG)$ whose associated homology class $[z] \in H_2(BG)$ satisfies $e[z] = 1$. To do this I will define a homomorphism $f: \pi_1(M) \to G$, for M a closed surface. This will induce a map $f: M \to BG$ and then $[z]$ will be $f_*[M]$ for M the fundamental class of M and f_* the induced homomorphism $H_2(M) \to H_2(BG)$. Now there is an induced bundle over M, $\gamma(f)$, with structure group G and hence a corresponding Euler

Class $\chi(\gamma(f)) : H_2(M) \to Z$. I will show that $-\chi(\gamma(f))[M] = 1$ for the particular $[z]$ I construct.

Now every element of H_2 of any group G can be represented as a sum of classes of the form $g_*[X]$ where X is a closed orientable surface and g_* is induced by a homomorphism $g : \pi_1(X) \to G$. (See for example the construction of H_2 of a group given in [6].) As a consequence χ can be considered to be a homomorphism $H_2(BG) \to Z$. I will show that $e = -\chi$. To make this identification I will use J. Wood's algorithm for computing χ, [7].

CONSTRUCTION OF $[z]$. Consider SL(2, **R**) as a subgroup of G by letting it act on the set of unoriented lines in $\mathbf{R}^2$ which can be identified with S^1. Namely if

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, \mathbf{R})$$

then

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax + by \\ cx + dy \end{pmatrix}$$

and as a diffeomorphism takes the line with slope $z = y/x$ to the line with slope $cx + dy/ax + by = c + dz/a + bz$.

Let

$$Q = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad R = \begin{pmatrix} 2 & 0 \\ 0 & 1/\sqrt{2} \end{pmatrix}, \quad S = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}.$$

Consider the commutators $[Q, R] = A$ and $[S, R] = B$. Then

$$A = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}.$$

Furthermore

$$ABA = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

which as a diffeomorphism of S^1 is rotation by 180° and

$$(ABA)(ABA) = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

which as a diffeomorphism of S^1 is the identity.

The equation $ABAABA = 1$ determines a homomorphism $f : \pi_1(M) \to G$ where M is a surface of genus 6. To construct $f_*[M]$ consider the chains

$$c_1 = (Q^{-1}R^{-1}, R^{-1}Q^{-1}) - (R^{-1}, Q) + (Q^{-1}, R) + (1, R) + (1, S) - (1, 1),$$

$$c_2 = (S^{-1}R^{-1}, R^{-1}S^{-1}) - (R^{-1}, S) + (S^{-1}, R) + (1, R) + (1, S) - (1, 1),$$

$$c_3 = (A, AB) + (AB, ABA) - (1, B) - (1, A),$$

$$\partial c_1 = A, \quad \partial c_2 = B$$

and

$$\partial c_3 = A + B + A - ABA = 2A + B - ABA.$$

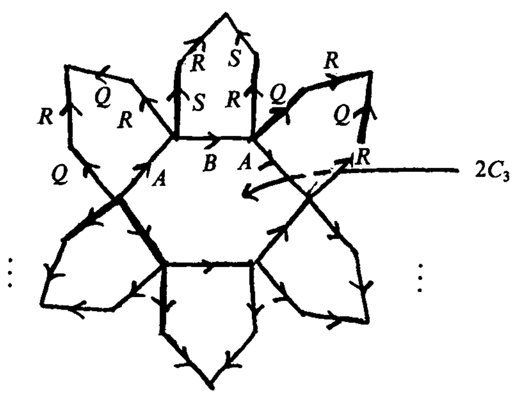
Now

$$c_4 = 2(c_1 + c_2 + c_1 - c_3) = 4c_1 + 2c_2 - 2c_3$$

satisfies $\partial c_4 = 2(ABA) = 2T$ where T is rotation by 180°. Set

$$z = -(c_4 - (1, T) - (1, 1)).$$

Then z is a cycle which is $f_*[M]$ as the following diagram easily shows.

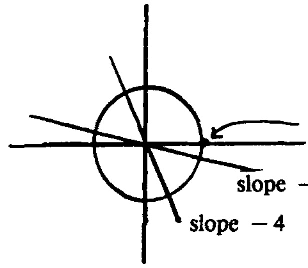


Now to compute $e[z]$ note that all of the 2-simplices making up this cycle except $(S^{-1}R^{-1}, R^{-1}S^{-1})$ of c_2 have at least one face with a fixed point at $[0] \in \mathbf{R}/Z$. So $e = 0$ on all these simplices. On the other hand

$$S^{-1}R^{-1} = \begin{pmatrix} 1/\sqrt{2} & 0 \\ -2/\sqrt{2} & \sqrt{2} \end{pmatrix} \quad \text{which is } z \to 2z - \tfrac{1}{2}$$

and

$$R^{-1}S^{-1} = \begin{pmatrix} 1/\sqrt{2} & 0 \\ -2/\sqrt{2} & \sqrt{2} \end{pmatrix} \quad \text{which is } z \to 2z - 4.$$



This means $\widetilde{S^{-1}}\widetilde{R^{-1}}(0) > \widetilde{R^{-1}}\widetilde{S^{-1}}(0)$.
So $e(S^{-1}R^{-1}, R^{-1}S^{-1}) = -\frac{1}{2}$ and
$e[z] = 1$ as claimed.

Theorem 2.
(a) $e: H_2(BG) \to Z$ *is an epimorphism.*
(b) $e = E$ *where* E *is the invariant of* [3].
(c) $e = -\chi$.

Proof. The construction of z above proves (a). For (b) I will make use of the constructions in [3]. E is the homomorphism induced on H_2 by the projection of BG on the "component complex" $|\pi_*|$ (see [3], §2), followed by the isomorphism

$$H_2(|\pi_*|) \tilde{\to} Z \qquad ([3], \text{ lemma } 5);$$

that is,

$$E: H_2(BG) \xrightarrow{P} H_2(|\pi_*|) \tilde{\to} Z.$$

The 2-cycle $\alpha = (\frac{1}{3}, \frac{2}{3}) - (\frac{2}{3}, \frac{1}{3})$ is a generator of $H_2(|\pi_*|)$ which maps to 1 under the isomorphism. Now a straightforward calculation shows that

$$P([z]) = 2 \cdot (\tfrac{2}{3}, \tfrac{1}{3}) - (\tfrac{1}{3}, 0) - (0, \tfrac{1}{3}).$$

Furthermore $P([z])$ is homologous to α in $|\pi_*|$:

$$\partial(\tfrac{2}{3}, 0, \tfrac{1}{2}) = (\tfrac{1}{3}, \tfrac{2}{3}) + (\tfrac{2}{3}, \tfrac{1}{3}) - (0, \tfrac{1}{3}) - (\tfrac{1}{3}, 0)$$

gives $P(z) - \partial(\frac{2}{3}, 0, \frac{1}{3}) = \alpha$.

So $E([z]) = E([\alpha]) = 1 = e([z])$.

I have proved that e and E agree on the homology class $[z] \in H_2(BG)$. The following argument will show they agree on all of $H_2(BG)$.

The construction of z defines a left inverse for E. So the main theorem of [3] becomes

$$H_2(G) \cong Z \oplus H_2(H)$$

and any homology class $[w] \in H_2(G)$ can be uniquely written

$$[w] = n \cdot E([z]) + [w] \qquad \text{for } n \in Z, \quad [w] \in H_2(H).$$

$e = E$ on $([z])$, hence on $[w]$ and this proves (b).

NOTE. J. Mather has proved that $H_2(H^0) = 0$, [4], so in the C^0 case e and E give isomorphisms $H_2(BG) \to Z$. On the other hand, the existence of the Godbillon–Vey invariant implies that $H_2(B^r) \neq 0$ for $r > 2$.

To prove (c) I will identify e with the invariant W of J. Wood (see lemma 2.1 of [7]) which is actually an algorithm for computing χ. W is easy to compute for the homomorphism $\pi_1(M) \to G$ defined by $ABAABA = 1$.

The lifts $\tilde{A}$ and $\tilde{B}$ of A and B have fixed points at $0 \in \mathbf{R}$, $\tilde{B}$ satisfies $\tilde{B}(0) \in (0, \frac{1}{2})$. So $\tilde{A}\tilde{B}\tilde{A}(z) = z + \frac{1}{2}$. Then $W = 1$. Now $W = -\chi$, which is the algorithm of Wood, and this proves (c).

3. A bound for the Euler Class

I will begin this section by stating that part of the theorem of J. Wood ([7], theorem 1.1) which gives an obstruction to reducing the structure group of a circle bundle over a closed surface to a discrete one.

Here is the set up. Let G be any of the groups G^r and let G_{top} be the group G with the C^r topology. Suppose there is given an oriented S^1 bundle γ with structure group G_{top} over a closed oriented surface X. This is equivalent to being given a classifying map $\gamma : X \to BG_{\text{top}}$ (defined up to homotopy). It is well known that up to homotopy $BG_{\text{top}} = BS^1 = K(Z, 2)$ so there is an isomorphism

$$\chi : H_2(BG_{\text{top}}) \to Z$$

which can be thought of as the universal Euler class.

Now if the given G_{top} bundle reduces to G there is a classifying map $g\gamma$: $X \to BG$ and a commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{\ \gamma\ } & BG_{\text{top}} \\ & \searrow^{g\gamma} & \uparrow \\ & & BG \end{array}$$

where the vertical arrow is induced by the identity $G \to G_{\text{top}}$.

On homology the following commutes:

$$\begin{array}{ccccc} H_2(X) & \xrightarrow{\gamma_*} & H_2(BG_{\mathrm{top}}) & \longrightarrow & Z \\ & {\scriptstyle g\gamma_*}\searrow & & \nearrow{\scriptstyle e} & \\ & & H_2(G) & & \end{array}$$

(Recall for a discrete group H the notation is $H_2(H) = H_2(BH)$ which is $H_2(K(H, 1))$.) Let $\varepsilon(X)$ stand for the Euler characteristic of X and $[X]$ for the fundamental class of X.

THEOREM 3 (Wood). *If* $\gamma : X \to BG_{\mathrm{top}}$ *factors through* $g\gamma : X \to BG$ *then* $|eg\gamma_*[X]| \leqq \varepsilon(X)$.

4. Complexity and simplicial versions of Wood's theorem

I will prove versions of Wood's theorem for any connected topological space. Consider the subcomplex of singular chains on X, $\hat{C}_*(X)$, consisting of those simplices mapping all vertices to a given base point of X. We can define a "norm" $\|\cdot\|_Z$ on $\hat{C}_*(X)$ by $\|\alpha\|_Z = \Sigma |m_i|$ where $c = \Sigma\, m_i\sigma_i$. Then on $H_*(X)$ define

$$\|\alpha\|_Z = \inf(\|c\|_Z,\ c \text{ is a cycle representing } \alpha).$$

This is a restriction of Gromov's norm; it is the smallest number of 2-cells needed to build α, counted according to their multiplicities. Note that the simplicial set $\hat{S}(X) \subset S(X)$ of singular simplices mapping vertices to a base point is weakly homotopy equivalent to the full singular complex so that the homology of $\hat{C}_*(X)$ computes $H_*(X)$.

THEOREM 4. *Let X be a connected topological space and $\alpha \in H_2(X)$ an arbitrary homology class. Let $g : X \to BG$ be any continuous map. Then*

$$|eg_*\alpha| \leqq \|\alpha\|_Z/2.$$

PROOF. First note that since BG is a $K(G, 1)$ there is a 1-1 correspondence between $[X, BG]$ and $\mathrm{Hom}(\pi_1(X), BG)$. On the other hand $\pi_1(X)$ is given by the free group on 1-simplices of $\hat{S}(X)$ modulo relations coming from 2-simplices, so each element of $\mathrm{Hom}(\pi_1(X), BG)$ induces a simplicial map from $\hat{S}(X)$ to BG and conversely. Hence there is a 1-1 correspondence between homotopy classes of continuous maps $[X, BG]$ and homotopy classes of simplicial maps $[\hat{S}(X), NG]_s$.

In particular the map g in the statement of the theorem is homotopic to the map induced on realizations by a simplicial map $\hat{g}$ from $S(X)$ to NG. Let

$z = \Sigma\, m_i z_i$ be a cycle representing α so that $|m_1| + \cdots + |m_n|$ is as small as possible. Then

$$|eg_*\alpha| = |\Sigma m_i e\hat{g}_*(z_i)| = \Sigma |m_i eg_*(z_i)| \leqq \Sigma |m_i|\,|eg_*(z_i)|.$$

Each element $eg_*(z_i)$ is $\frac{1}{2}$, $-\frac{1}{2}$, or 0. Therefore

$$|eg_*\alpha| \leqq \Sigma |m_i|/2 = \|\alpha\|/2.$$

This proves Theorem 4.

The complexity of a circle bundle. The statement of Theorem 4 would be more satisfactory if there were a single "test element" β so that if the inequality $|eg_*\beta| \leqq \|\beta\|/2$ holds for β then it holds for all $\alpha \in H_2(X)$. To make such a choice possible requires passing to real homology.

Consider the real singular chain complex, but constructed only out of simplices mapping all vertices to a base point. Let $H_*(X, \mathbf{R})$ be the associated homology groups. The homomorphism e extends to a homomorphism $e_{\mathbf{R}}: H_2(X, \mathbf{R}) \to \mathbf{R}$. For any $\alpha \in H_2(X, \mathbf{R})$ there is the Gromov norm, constructed as before, $\|\alpha\|$.

I will define an invariant $\kappa(\gamma)$ of a circle bundle γ over X called the *complexity* of γ. Consider the affine subspace $e_{\mathbf{R}}^{-1}(1) \subset H_2(X, \mathbf{R})$, assuming it is not empty. Let $\kappa(\gamma)$ be the $\inf\{\|z\|, z \in e_{\mathbf{R}}^{-1}(1)\}$.

THEOREM 5. *Let X be a connected topological space and γ a circle bundle over X. Suppose γ reduces to a bundle with discrete structure group G. Then $\kappa(\gamma) \geqq 2$.*

PROOF. Theorem 4, adapted to real homology, gives

$$\|\alpha\|/|eh_*\alpha| \geqq 2$$

where h is the classifying map of γ and α is any element of $H_2(X, \mathbf{R})$. Or

$$\|\alpha/eh_*\alpha\| \geqq 2 \qquad \text{for all } \alpha \in H_2(X, \mathbf{R}).$$

In particular, any element in $e_{\mathbf{R}}^{-1}(1)$ satisfies this inequality, and this proves the theorem.

5. An application to foliations

THEOREM 6. *Assume that X is a differentiable manifold. Let γ be a circle bundle over X and let $\kappa(\gamma)$ be its complexity. If $\kappa(\gamma) < 2$ then there is no codimension-1 foliation on the total space of γ transverse to the fibers.*

This follows directly from Theorem 5 and the fact that foliating the total space of γ transverse to the fibers is equivalent to factoring the classifying map of γ through BG; see [7].

REFERENCES

1. E. Ghys, *Groupes d'homeomorphismes du cercle et cohomologie bornee*, preprint.
2. M. Gromov, *Volume and bounded cohomology*, Publ. Math. I.H.E.S. **56** (1982), 5–100.
3. S. Jekel, *On the second homology group of the discrete group of diffeomorphisms of the circle*, preprint.
4. J. Mather, *The vanishing of the homology of certain groups of homeomorphisms*, Topology **10** (1971), 297–298.
5. J. P. May, *Simplicial Objects in Algebraic Topology*, Van Nostrand Mathematical Studies #11, Princeton, 1967.
6. C. Miller, *The second homology group of a group*, Proc. Am. Math. Soc. 3 (1952), 588–595.
7. J. Wood, *Bundles with totally disconnected structure group*, Comm. Math. Helv. **46** (1971), 257–273.